

Optimal control results for semilinear integro-differential stochastic systems via resolvent operators

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Abstract

This paper presents optimal control results for second-order semilinear integro-differential stochastic systems in Hilbert space. Using Banach's fixed point theorem, we derive the existence and uniqueness of the mild solution. The optimal control is demonstrated through the minimizing sequence approach and Mazur's lemma. Our approach integrates resolvent operator theory to manage the system's integral and stochastic components effectively. We derive necessary and sufficient optimality conditions and validate our theoretical results through an illustrative example.

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1 Introduction

Optimal control theory for dynamical systems involves determining control inputs that optimize a performance criterion while steering the system from an initial to a desired state. Examples include designing efficient flight paths for aircraft to minimize fuel consumption, optimizing investment portfolios to maximize returns while managing risk, and developing control strategies for automated manufacturing processes to enhance productivity. These applications demonstrate how optimal control theory can improve the performance, stability, and efficiency of various dynamic systems across different fields.

The work of Balakrishnan, described in [1], systematically examines optimal control problems in the context of infinite-dimensional semilinear systems. Additionally, the study uses state and control variables in Banach spaces and relies on the concept of bounded resolvent to obtain results. Expanding on the semicontinuity

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of integral functions at the lower end, Papageorgiou [2] builds on the seminal work of Casting and Clauzure. This study examines the existence of optimal controls in Banach spaces for nonlinear systems across various system classes, imposing specific conditions for optimal control existence. Papageorgiou also makes a contribution in [3], presenting the essential "bang-bang characterisation" and demonstrating sufficient optimality criteria using the penalty function approach. An unconventional method, as seen in [4], addresses a system with non-monotone nonlinearities driven by evolution equations, but without limitations. The theoretical basis for linear retarded control systems is expanded by Park et al.'s study in [5], which focuses on essential optimality requirements for these systems. Furthermore, by utilizing the Cesari principle and the Fillippov theorem, the authors of [6] show that there are workable solutions to the Lagrange problem. In [7], Park et al. investigate this domain by applying these results to delay differential equations. Jeong et al. investigate Lipschitz nonlinearity with and without delay in a series of works [8–10] addressing semilinear evolution equations. By easing essential differentiability constraints for the nonlinear component, their contributions analyze the existence of optimal control and create the maximal principle. Finally, writers utilize their study for two practical applications in [9] one that deals with terminal values and the other focusing on control strategy averaging.

As noted in references [11–15], Grimmer played a key role in developing resolvent operators for the investigation and demonstration of integro-differential systems. Combining the usage of the resolvent operator with fixed-point techniques proves to be a straightforward and very efficient strategy when working with integro-differential equations. We highly suggest reading [11–17] and the related literature for a more comprehensive understand of resolvent operators. The author has presented the idea of nonlocal analytic resolvent operator integro-differential equations in recent publications [16, 17]. This invention makes use of the regularity of development in integro-differential systems and insights into resolvent operators. Moreover, the author explores the controllability of integro-differential inclusions using resolvent operators in a fairly recent work [18], utilizing resolvent operators and Bohnenblust-Karlin's fixed-point method.

Wang et al. (as described in [19]) addressed the idea of mild solutions in the context of fractional-order systems, using a priori estimation approaches and extending these solutions to a global interval within integro-differential fractional systems. In reflexive Banach spaces with specific delays, their most recent studies [20–24] investigate the necessary conditions for the best possible management of abstract fractional semilinear systems. They used well-established methods to obtain conclusions for both stochastic and deterministic systems. Furthermore, these authors investigated the best control results for real-world nonlinear control problems such as dispersion equations, population dynamics, and thermoelasticity, in [25–27]. These studies highlight the relevance of their findings by highlighting optimal control strategies in real-world situations.

The foundational concepts introduced in references [28] and [29], incorporating a fixed-point theorem to establish conditions ensuring the existence and uniqueness

of the mild solution. Using resolvent operators, the author of [30] derives sufficient conditions for optimal control in second-order semilinear integro-differential control systems. Moreover, the authors extend their exploration to the realm of optimal control for this semilinear second-order system. In [31],[32] authors investigate optimal feedback control for stochastic fractional systems in Hilbert spaces, focusing on neutral and Caputo fractional evolution systems. Employing fractional calculus and fixed point theory, it establishes the existence of mild solutions and feasible pairs through advanced mathematical techniques. Recently author [33] explores optimal control of fractional stochastic nonlocal semilinear systems in Hilbert space, establishing existence and uniqueness of mild solutions and proving optimal control using advanced mathematical techniques. To the best of our knowledge, no paper has been published on the optimal control for stochastic second-order semilinear integro-differential systems employing resolvent operators, which is motivated by the work mentioned above. Our results are based on the fundamental concepts of [30–39] with appropriate adjustments. These findings mark a significant leap forward, contributing substantially to the advancement of optimal control theory specifically tailored for second-order systems. This innovative approach offers new perspectives and methodologies, potentially enhancing the understanding of these complex control systems.

The rest of the paper is structured as follows:

Segment 2: Introduces elementary definitions crucial to understanding the subsequent discussions.

Segment 3: Focuses on proving the existence and uniqueness of the mild solution (with Lipschitz nonlinearity) for the given control differential equation.

Segment 4: Explores and establishes the existence of optimal control within the context of the presented framework.

Segment 5: Provides an illustrative example aimed at validating and showcasing the practical implications of the obtained results.

2 Preliminaries

Consider two separable Hilbert spaces U and V . Let I represents the interval $[0, T]$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ represents a complete probability space, with $\mathcal{F}_\varsigma \subset \mathcal{F}$ and $\mathcal{F}_\varsigma, 0 \leq \varsigma \leq T$, which are right continuous increasing it contains a complete family of sub σ -algebras. $\{e_n\}_{n=1}^\infty$ represents the complete orthonormal system in U and $\{\beta_n\}_{n=1}^\infty$ represents a sequence of independent Brownian motions which satisfy the following condition.

$$W(\varsigma) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n e_n, \quad 0 \leq \varsigma \leq T.$$

where $\{\lambda_n\}_{n=1}^\infty$ is a bounded sequence and $\lambda_n \in \mathbb{R}^+ \cup \{0\}$ for $n \in \mathbb{N}$, with $Qe_n = \lambda_n e_n$, $n \in \mathbb{N}$ with $\text{tr}(Q) = \sum_{n=1}^\infty \lambda_n < \infty$. Then, a U -valued stochastic process $W(\varsigma)$ is referred to as a Wiener Process. $\mathcal{F}_\varsigma$ represents the sigma algebra that is

produced by $\{W(z) : 0 \leq z \leq \varsigma\}$, and it is referred to as normal filtration. $F_T = F$ as well.

Let $L(U, V)$ represent the space of bounded operators defined from U to V , using the usual operator norm. For $\psi \in L(U, V)$, we define

$$\|\psi\|_Q^2 = \text{tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2.$$

$\|\psi\|_Q^2 < \infty$, then ψ is a Q -Hilbert Schmidt operator. $L_Q(U, V)$ represents the space of operators $\psi : U \rightarrow V$, where ψ is the Q -Hilbert Schmidt operator. $L_Q(U, V)$ represents a completion of $L(U, V)$ with respect to the topology created by the norm $\|\cdot\|_Q$ is a Hilbert space related to the topology of norms.

In terms of norm topology $\|z(\cdot)\| = (\mathbb{E}\|z(\varsigma)\|^2)^{1/2}$, $L_2(\Omega, V)$ is a Banach space that contains V -valued integrable, strongly measurable random variables, where $\mathbb{E}(\cdot)$ represents the expectation with respect to the measure $\mathbb{P}$.

Let $C([0, T], L_2(\Omega, V))$ be the space of continuous maps defined from $[0, T]$ into $L_2(\Omega, V)$, which is a Banach space and fulfilling $\sup_{0 \leq \varsigma \leq T} \mathbb{E}\|z(\varsigma)\|^2 < \infty$. Suppose V_2 is the closed subspace of $C([0, T], L_2(\Omega, V))$ with F_ς -adapted, measurable, V -valued processes $z \in C([0, T], L_2(\Omega, V))$ equipped with the norm

$$\|z\|_{V_2} = \left(\sup_{0 \leq \varsigma \leq T} \mathbb{E}\|z(\varsigma)\|_V^2 \right)^{\frac{1}{2}}.$$

Let the integral cost function be

$$(1) \quad J(z, v) := \mathbb{E} \left\{ \int_0^T L(\varsigma, z^v(\varsigma), v(\varsigma)) d\varsigma \right\},$$

subject to the equations

$$z''(\varsigma) = A(\varsigma)z(\varsigma) + \int_0^\varsigma \mathcal{B}(\varsigma, \omega)z(\omega)d\omega + Bv(\varsigma) + F(\varsigma, z(\varsigma)) + \sigma(\varsigma, z(\varsigma))\frac{dW(\varsigma)}{d\varsigma},$$

$$\varsigma \in I = [0, T],$$

$$z(0) = z_0 \in V,$$

$$(2) \quad z'(0) = z_1 \in V.$$

In section 4, the integrand L is defined, the function $z(\cdot)$ is V -valued stochastic process. The domain U of the control function $v(\cdot)$ is a separable reflexive Hilbert space. The operators $A(\varsigma)$ and $\mathcal{B}(\varsigma, \omega)$, defined respectively on domains $D(A(\varsigma)) \subseteq V$ and $D(\mathcal{B}) \subseteq V$, are closed and linear within the space V . Furthermore, the operator $B : U \rightarrow V$ is bounded, the functions $F : I \times V \rightarrow V$ and $\sigma : [0, T] \times V \rightarrow L(U, V)$ are nonlinear. $z(0)$, z' are V -valued F_0 -measurable random variables which are independent of W . Notably, the domain $D(\mathcal{B})$ remains independent of the parameters (ς, ω) in consideration.

The admissible set U_{ad} is defined as follows:

$$U_{ad} := \{v(\cdot) | [0, T] \times \Omega \rightarrow U \text{ is measurable}\}$$

as per [40], v represents a stochastic process included to F_ς , and $\mathbb{E} \int_0^T \|v(\varsigma)\|^p d\varsigma < \infty$. It is evident that $U_{ad} \subset L_p([0, T]; U)$ ($1 < p < \infty$) and $U_{ad} \neq \emptyset$ are convex, closed, and bounded.

Consider $\mathcal{A}_{ad}$ denote the set of admissible state-control pairs (z, v) , z denotes the mild solution of a semilinear differential equation (2) subject to the control $v \in U_{ad}$. The optimal control problem is formulated as:

Find $(z^0, v^0) \in \mathcal{A}_{ad}$ such that

$$J(z^0, v^0) := \inf J(z, v) : (z, v) \in \mathcal{A}_{ad} = \delta.$$

Hereafter, the essential findings, fundamental theories, and crucial lemmas are presented systematically. The subsequent content unfolds as follows, $(V, |\cdot|)$ signifies a Banach space. The operators $A(\varsigma)$ and $\mathcal{B}(\varsigma, \omega)$, where $0 \leq \omega \leq \varsigma$, represent linear operators, which is closed, established on $D(A)$ and $D(\mathcal{B})$, respectively. An essential assumption is that the density of $D(A)$ should be contained in V . $D(A)$ is the space equipped with the graph norm produced by $A(\varsigma)$ constitutes a Banach space. For simplicity, it's assumed that all these norms are interchangeable. A straightforward condition ensuring this characteristic is the existence of $\lambda \in \rho(A(\varsigma))$, the resolvent set of $A(\varsigma)$, such that a bounded linear operator $(\lambda I - A(\varsigma))^{-1} : V \rightarrow D(A)$ exists.

Consider the second-order IVP,

$$(3) \quad z''(\varsigma) = A(\varsigma)z(\varsigma) + f(\varsigma), \quad 0 \leq \varsigma \leq T,$$

$$(4) \quad z(\omega) = z^0, \quad z'(\omega) = z^1.$$

In the given setup, $A(\varsigma) : D(A) \subseteq V \rightarrow V$, where $\varsigma \in [0, T]$, stands as a closed linear operator with dense definition. Furthermore, $f : [0, T] \rightarrow V$ represents a fitting function. For more comprehensive insights, readers are directed to explore references [29, 41–45]. Many of these sources extensively discuss the relationship between the existence of equations (3)-(4) and the form of $S(\varsigma, s)$, represented as

$$z''(\varsigma) = A(\varsigma)z(\varsigma), \quad 0 \leq \varsigma \leq T.$$

If we suppose that $\forall z \in D(A)$, the mapping $\varsigma \mapsto A(\varsigma)z$ is continuous, we then assume that $A(\cdot)$ generates the sequence $(S(\varsigma, \omega))_{0 \leq \omega \leq \varsigma \leq T}$. This concept is elaborated in Kozak's work [42], under Remark 2.1, and is also referenced in Henríquez's work [46], specifically in Definition (1.1). We recommend consulting these works for a detailed examination of this matter. However, in our present scope, we consider that $S(\cdot)z$ exhibits continuous differentiability $\forall z \in V$, with its derivative uniformly bounded on bounded intervals. This condition suggests the existence of $K_1 > 0$ such that

$$\|S(\varsigma + h, \omega) - S(\varsigma, \omega)\| \leq K_1|h|,$$

$\forall \omega, \varsigma, \varsigma + h \in [0, T]$. Let's determine the operator $C(\varsigma, \omega) = -\frac{\partial S(\varsigma, \omega)}{\partial \omega}$. Suppose $f : [0, T] \rightarrow V$ is an integrable function. The mild solution can be defined, $z :$

$[0, T] \rightarrow V$ of the system (3)-(4) as:

$$z(\varsigma) = C(\varsigma, \omega)z^0 + S(\varsigma, \omega)z^1 + \int_0^\varsigma S(\varsigma, \xi)f(\xi)d\xi.$$

Introduce the integro-differential system of second order

$$(5) \quad z''(\varsigma) = A(\varsigma)z(\varsigma) + \int_\omega^\varsigma \mathcal{B}(\varsigma, \xi)z(\xi)d\xi, \quad \omega \leq \varsigma \leq T,$$

$$(6) \quad z(\omega) = 0, \quad z'(\omega) = x \in V,$$

for $0 \leq \omega \leq T$. This problem has been addressed in [45]. Let us denote the set $\Delta = \{(\varsigma, \omega) : 0 \leq \omega \leq \varsigma \leq T\}$. Now, let us introduce certain conditions that the operator $\mathcal{B}(\cdot)$ satisfies:

Remark 2.1. [45]

(B1) For each $0 \leq \omega \leq \varsigma \leq T$, $\mathcal{B}(\varsigma, \omega) : [D(A)] \rightarrow V$ is a bounded linear operator, additionally for every $z \in D(A)$, $\mathcal{B}(\cdot, \cdot)z$ is continuous and

$$\|\mathcal{B}(\varsigma, \omega)z\| \leq b\|z\|_{[D(A)]},$$

for $b > 0$ which is independent of $\omega, \varsigma \in \Delta$.

(B2) There exists $L_{\mathcal{B}} > 0$ such that

$$\|\mathcal{B}(\varsigma_2, \omega)z - \mathcal{B}(\varsigma_1, \omega)z\| \leq L_{\mathcal{B}}|\varsigma_2 - \varsigma_1|\|z\|_{[D(A)]},$$

$\forall z \in D(A)$, $0 \leq \omega \leq \varsigma_1 \leq \varsigma_2 \leq T$.

(B3) There exists $b_1 > 0$ such that

$$\left\| \int_0^\varsigma S(\varsigma, \omega)\mathcal{B}(\omega, \varsigma)z d\omega \right\| \leq b_1\|z\|$$

for all $z \in D(A)$.

It has been demonstrated that a family $(\mathcal{V}(\varsigma, \omega))_{\varsigma \geq \omega}$ is associated with the problem (5)-(6) under these circumstances. Going forward, Assuming the presence of a resolvent operator, let's include its features into our definition.

Definition 2.2. [45] "A family of bounded linear operators $(\mathcal{V}(\varsigma, \omega))_{\varsigma \geq \omega}$ on V is called a resolvent operator for the system (5)-(6) if it satisfies:

- (a) The map $\mathcal{V} : \Delta \rightarrow \mathcal{L}(V)$ is strongly continuous, $\mathcal{V}(\varsigma, \cdot)z$ is continuously differentiable for all $z \in V$, $\mathcal{V}(\omega, \omega) = 0$, $\frac{\partial}{\partial \varsigma}\mathcal{V}(\varsigma, \omega)|_{\varsigma=\omega} = I$ and $\frac{\partial}{\partial \omega}\mathcal{V}(\varsigma, \omega)|_{\omega=\varsigma} = -I$.
- (b) For any $(\varsigma, \omega) \in \Delta$ and for any $x \in D(\varsigma, \omega)$, the function $\mathcal{V}(\varsigma, \omega)x$ is a solution of the system (5)-(6), which means that

$$\frac{\partial^2}{\partial \varsigma^2}\mathcal{V}(\varsigma, \omega)x = A(\varsigma)\mathcal{V}(\varsigma, \omega)x + \int_\omega^\varsigma \mathcal{B}(\varsigma, \xi)\mathcal{V}(\xi, \omega)x d\xi.$$

for all $0 \leq \omega \leq \varsigma \leq T$.

Condition (a) implies the existence of $K > 0$ and $\tilde{K} > 0$ such that

$$\|\mathcal{V}(\varsigma, \omega)\| \leq K, \quad \left\| \frac{\partial}{\partial \varsigma} \mathcal{V}(\varsigma, \omega) \right\| \leq \tilde{K}, \quad (\varsigma, \omega) \in \Delta.$$

Furthermore, the operator that is linear

$$G(\varsigma, \zeta)x = \int_{\zeta}^{\varsigma} \mathcal{B}(\varsigma, \omega) \mathcal{V}(\omega, \varsigma) x d\omega, \quad x \in D(A), \quad 0 \leq \zeta \leq \varsigma \leq T,$$

can be extended to the space V . Denoting this extension by the same notation $G(\varsigma, \zeta)$, the mapping $G : \Delta \rightarrow \mathcal{L}(V)$ is strongly continuous. Moreover, it can be verified

$$(7) \quad \mathcal{V}(\varsigma, \zeta)x = S(\varsigma, \zeta) + \int_{\zeta}^{\varsigma} S(\varsigma, \omega) G(\omega, \zeta) x d\omega, \quad \forall x \in V.$$

As $\mathcal{V}(\cdot)$ is uniformly Lipschitz continuous, there exists $L_V > 0$ such that

$$(8) \quad \|\mathcal{V}(\varsigma + h, \zeta) - \mathcal{V}(\varsigma, \zeta)\| \leq L_V |h|, \quad \forall \varsigma, \varsigma + h, \zeta \in [0, T].$$

Let an integrable function $g : I \rightarrow V$. The following nonhomogeneous problem

$$(9) \quad z''(\varsigma) = A(\varsigma)z(\varsigma) + \int_0^{\varsigma} \mathcal{B}(\varsigma, \omega) z(\omega) d\omega + g(\varsigma), \quad \varsigma \in I = [0, T],$$

$$(10) \quad z(0) = x^0, \quad z'(0) = x^1,$$

has been discussed in [45]. Now, let's present the mild solution for the system (9)-(10).

Definition 2.3. [45] "Assume $x^0, x^1 \in V$. The continuous function $z : [0, T] \rightarrow V$ given by

$$z(\varsigma) = -\frac{\partial \mathcal{V}(\varsigma, 0)x^0}{\partial \varsigma} + \mathcal{V}(\varsigma, 0)x^1 + \int_0^{\varsigma} \mathcal{V}(\varsigma, \omega) g(\omega) d\omega$$

is called the mild solution for the system (9)-(10)."

Definition 2.4. A mild solution of system (2) is defined as a F_{ς} -adapted stochastic process $z(\varsigma) \in C([0, T]; L_2(\Omega, V))$ if $z(\varsigma)$ is measurable for each $v(\cdot) \in L_p([0, T]; U)$ and the following stochastic integral equation:

$$(11) \quad \begin{aligned} z(\varsigma) = & -\frac{\partial \mathcal{V}(\varsigma, 0)}{\partial \varsigma} z_0 + \mathcal{V}(\varsigma, 0) z_1 + \int_0^{\varsigma} \mathcal{V}(\varsigma, \omega) B v(\omega) d\omega + \int_0^{\varsigma} \mathcal{V}(\varsigma, \omega) F(\omega, z(\omega)) d\omega \\ & + \int_0^{\varsigma} \mathcal{V}(\varsigma, \omega) \sigma(\omega, z(\omega)) dW(\omega), \quad \varsigma \in I, \end{aligned}$$

is fulfilled.

3 Mild Solution

In this section we discuss the existence and uniqueness of mild solution for equation (2). Some constraints on nonlinear functions are applied in order to get the results.

- (i) The functions $\varsigma \rightarrow F(\varsigma, z(\varsigma))$ and $\varsigma \rightarrow \sigma(\varsigma, z(\varsigma))$ are measurable for all $z \in V$.
- (ii) $F : [0, T] \times V \rightarrow V$, $\sigma : [0, T] \times V \rightarrow L(U, V)$ are functions that satisfy both the linear growth and Lipschitz conditions. Furthermore, F & σ are continuous functions. Overall, it is assumed that there are constants that are positive L_1 and L_2 such that

$$\begin{aligned} \|F(t, z) - F(t, w)\|^2 &\leq L_1 \|z - w\|^2, & \|F(t, z)\|^2 &\leq L_F(1 + \|z\|^2), \\ \|\sigma(t, z) - \sigma(t, w)\|^2 &\leq L_2 \|z - w\|^2, & \|\sigma(t, z)\|^2 &\leq L_\sigma(1 + \|z\|^2). \end{aligned}$$

- (iii) Consider a control problem where the controls, denoted v are chosen from a Hilbert space that is reflexive U . The operator $B \in L_\infty([0, T]; L(U, V))$ and $\|B\|_\infty$ stands for the norm of operator B in the Banach space $L_\infty([0, T]; L(U, V))$. $L(U, V)$ represents all operators that have been defined U to V that are both linear and bounded.
- (iv) The map $\tilde{U}(\cdot) : [0, T] \rightrightarrows 2^{\tilde{U}} \setminus \{\phi\}$ signifies a multivalued operator with closed, convex, and bounded value sets, where $\tilde{U}(\cdot) \subseteq \Omega$ is graph measurable for bounded set Ω in $\tilde{U}$. Here $2^{\tilde{U}}$ represents the power set of $\tilde{U}$.

Theorem 3.1. *For any control function $v(\cdot) \in U_{ad}$, the system (2) exhibits a unique mild solution in $[0, T]$, under the provided assumptions (i) – (iv).*

Proof: Define an operator $\Phi : V_2 \rightarrow V_2$ such that

$$\begin{aligned} (\Phi z)(\varsigma) &= -\frac{\partial}{\partial \varsigma} \mathcal{V}(\varsigma, 0)z_0 + \mathcal{V}(\varsigma, 0)z_1 + \int_0^\varsigma \mathcal{V}(\varsigma, \omega)Bv(\omega)d\omega + \int_0^\varsigma \mathcal{V}(\varsigma, \omega)F(\omega, z(\omega))d\omega \\ &+ \int_0^\varsigma \mathcal{V}(\varsigma, \omega)\sigma(\omega, z(\omega))dW(\omega). \end{aligned}$$

We shall now establish the result that provides a mild solution (11) for the system (2) on $[0, T]$. To do this, we shall demonstrate that Φ has a fixed point in space V_2 . The result is demonstrated using the classical contraction fixed point theorem. Firstly, we prove that $\Phi(V_2) \subset V_2$. Let $z \in V_2$, then

$$(12) \quad \mathbb{E} \|(\Phi z)(\varsigma)\|^2 \leq 5[T_1 + T_2 + T_3 + T_4 + T_5]$$

Now

$$\begin{aligned} T_1 &= \mathbb{E} \left\| -\frac{\partial}{\partial \varsigma} \mathcal{V}(\varsigma, 0)z_0 \right\|^2 \\ &\leq 2\tilde{K}^2 \|z_0\|^2 \end{aligned}$$

Similarly $T_2 \leq 2K^2 \|z_1\|^2$. Next, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
 T_3 &= \mathbb{E} \left\| \int_0^\varsigma \mathcal{V}(\varsigma, \omega) B v(\omega) d\omega \right\|^2 \\
 &= \left(\int_0^\varsigma \|\mathcal{V}(\varsigma, \omega)\| \|B(\omega)\| \mathbb{E} \|v(\omega)\| d\omega \right)^2 \\
 &\leq K^2 \|B\|_\infty^2 \left[\int_0^\varsigma \mathbb{E} \|v(\omega)\|^2 d\omega \right]^2 \\
 &\leq K^2 \|B\|_\infty^2 \left[\left(\int_0^\varsigma d\omega \right)^{\frac{p-1}{p}} \left(\int_0^\varsigma \|v(\omega)\|_U^p d\omega \right)^{\frac{1}{p}} \right]^2 \\
 &\leq K^2 \|B\|_\infty^2 \|v\|_{L_p([0, T]; U)}^2 T^{\frac{2(p-1)}{p}}
 \end{aligned}$$

According to condition (ii) and the cauchy-Schwartz inequality,

$$\begin{aligned}
 T_4 &= \mathbb{E} \left\| \int_0^\varsigma \mathcal{V}(\varsigma, \omega) F(\omega, z(\omega)) d\omega \right\|_V^2 \\
 &\leq \mathbb{E} \left(\int_0^\varsigma \|\mathcal{V}(\varsigma, \omega) F(\omega, z(\omega))\|_V d\omega \right)^2 \\
 &\leq K^2 \mathbb{E} \left(\int_0^\varsigma \|F(\omega, z(\omega))\|_V d\omega \right)^2 \\
 &\leq K^2 T \int_0^\varsigma \mathbb{E} \|F(\omega, z(\omega))\|_V^2 d\omega \\
 &\leq K^2 T \int_0^\varsigma L_F (1 + \mathbb{E} \|z(\omega)\|_V^2) d\omega \\
 &\leq K^2 T L_F \int_0^\varsigma \left(1 + \sup_{\omega \in [0, T]} \mathbb{E} \|z(\omega)\|_V^2 \right) d\omega \\
 &\leq K^2 T L_F T (1 + \|z\|_{V_2}^2) \\
 &= K^2 T^2 L_F (1 + \|z\|_{V_2}^2)
 \end{aligned}$$

and

$$\begin{aligned}
 T_5 &= \mathbb{E} \left\| \int_0^\varsigma \mathcal{V}(\varsigma, \omega) \sigma(\omega, z(\omega)) dW(\omega) \right\|^2 \\
 &\leq \mathbb{E} \left(\int_0^\varsigma \|\mathcal{V}(\varsigma, \omega) \sigma(\omega, z(\omega))\| dW(\omega) \right)^2 \\
 &\leq K^2 \text{tr}(Q) T \left(\int_0^\varsigma \mathbb{E} \|\sigma(\omega, z(\omega))\|_Q^2 dW(\omega) \right) \\
 &\leq K^2 \text{tr}(Q) T L_\sigma T (1 + \|z\|_{V_2}^2) \\
 &= K^2 \text{tr}(Q) T^2 L_\sigma (1 + \|z\|_{V_2}^2)
 \end{aligned}$$

Thus (12) becomes

$$\mathbb{E} \|(\Phi z)(\varsigma)\|^2 \leq a + T \|z\|_{V_2}^2$$

where the preferred constants are $a > 0$ and $T > 0$. The outcome is that Φ maps V_2 into itself.

We then demonstrate that Φ is a contraction map. For $w_1, w_2 \in V_2$, applying the Cauchy-Schwartz inequality and condition (ii), it concludes

$$\begin{aligned} \|(\Phi w_1)(\varsigma) - (\Phi w_2)(\varsigma)\|^2 &\leq 2\mathbb{E}\left\|\int_0^\varsigma \mathcal{V}(\varsigma, \omega)[F(\varsigma, w_1(\nu)) - F(\varsigma, w_2(\nu))]d\nu\right\|^2 \\ &+ 2\mathbb{E}\left\|\int_0^\varsigma \mathcal{V}(\varsigma, \omega)[\sigma(\varsigma, w_1(\nu)) - \sigma(\varsigma, w_2(\nu))]dW(\nu)\right\|^2 \\ &\leq 2\left(K^2(L_1 + L_2 \text{tr}(Q))T^2\right)\|w_1 - w_2\|_{V_2}^2 \end{aligned}$$

Consequently if

$$(13) \quad 2\left(K^2(L_1 + L_2 \text{tr}(Q))T^2\right) < 1$$

therefore it is evident that Φ has a unique fixed point in V_2 , which is a solution of (2). Proceeding with the previously mentioned procedure on the interval $[0, T^*]$, $[T^*, 2T^*]$, ... so that T^* fulfills (13), The additional condition on T can be removed easily.

In order to get the main outcomes, for system (2), we construct a priori estimate of the mild solution.

Lemma 3.2. (A priori estimate) Suppose that system (11) on $[0, T]$ is the mild solution of system (2), corresponding to control v . Then, a constant $C = C(v) > 0$ exists such that

$$\mathbb{E}\|z(\varsigma)\|^2 \leq C, \forall \varsigma \in [0, T].$$

Proof: With the use of Hölder's inequality and conditions (ii), we obtain

$$\begin{aligned} \mathbb{E}\|z(\varsigma)\|^2 &\leq 5\mathbb{E}\left\|-\frac{\partial}{\partial \varsigma}\mathcal{V}(\varsigma, 0)z_0\right\|^2 + 5\mathbb{E}\|\mathcal{V}(\varsigma, 0)z_1\|^2 + 5\mathbb{E}\left\|\int_0^\varsigma \mathcal{V}(\varsigma, \omega)Bv(\omega)d\omega\right\|^2 \\ &+ 5\mathbb{E}\left\|\int_0^\varsigma \mathcal{V}(\varsigma, \omega)F(\omega, z(\omega))d\omega\right\|^2 + 5\mathbb{E}\left\|\int_0^\varsigma \mathcal{V}(\varsigma, \omega)\sigma(\omega, z(\omega))dW(\omega)\right\|^2 \\ &\leq 5\tilde{K}^2\|z_0\|^2 + 5K^2\|z_1\|^2 + 5K^2\|B\|_\infty^2\left[\int_0^\varsigma \|v(\omega)\|d\omega\right]^2 \\ &+ 5K^2(L_F + \text{tr}(Q)L_\sigma)T\int_0^\varsigma \{1 + \mathbb{E}\|z(\omega)\|^2\}d\omega \\ &\leq 5\tilde{K}^2\|z_0\|^2 + 5K^2\|z_1\|^2 + 5K^2\|B\|_\infty^2\left[\left(\int_0^\varsigma d\omega\right)^{\frac{p-1}{p}}\left(\int_0^\varsigma \|v(\omega)\|_U^p d\omega\right)^{\frac{1}{p}}\right]^2 \\ &+ 5K^2(L_F + \text{tr}(Q)L_\sigma)T\int_0^\varsigma \{1 + \mathbb{E}\|z(\omega)\|^2\}d\omega \\ &\leq 5\tilde{K}^2\|z_0\|^2 + 5K^2\|z_1\|^2 + 5K^2\|B\|_\infty^2\|v\|_{L_p([0, T]; U)}^2 T^{\frac{2(p-1)}{p}} \end{aligned}$$

$$+ 5K^2(L_F + \text{tr}(Q)L_\sigma)T + 5K^2(L_F + \text{tr}(Q)L_\sigma)T \int_0^\varsigma \mathbb{E} \|z(\omega)\|^2 d\omega.$$

It is now simple to determine the boundedness of z in V_2 by applying Gronwall's inequality.

4 Results for Optimal Control

This section discusses existence results for second order stochastic optimal control under the following assumptions:

Let us define the integrand as follows:

$$L : [0, T] \times V \times U \rightarrow \mathbb{R}^*.$$

where $\mathbb{R}^* = \mathbb{R} \cup \infty$, the following assumptions are satisfied by the integrand L :

(M1) $L : [0, T] \times V \times U \rightarrow \mathbb{R} \cup \{\infty\}$ is F_ς -measurable.

(M2) For all $z \in V$, $\varsigma \in [0, T]$, $L(\varsigma, z, \cdot)$ is convex on U .

(M3) For almost all $\varsigma \in [0, T]$, $L(\varsigma, \cdot, \cdot)$ is sequentially lower semicontinuous on $V \times U$.

(M4) $l \geq 0, j > 0, \alpha \geq 0$ and $\alpha \in L_1([0, T]; \mathbb{R})$ are constants that satisfy the following condition:

$$L(\varsigma, z, v) \geq \alpha(\varsigma) + l\mathbb{E}\|z\|^2 + j\mathbb{E}\|v\|_U^p$$

Theorem 4.1. *The assumptions regarding the existence of a mild solution along with conditions (M1) – (M4) being fulfilled imply for system (1), there is at least one optimal pair $(z^0, v^0) \in \mathcal{A}_{ad}$. This pair satisfies:*

$$J(z^0, v^0) := \mathbb{E} \left\{ \int_0^T L(\varsigma, z^0(\varsigma), v^0(\varsigma)) d\varsigma \right\} \leq J(z, v), \forall (z, v) \in \mathcal{A}_{ad}.$$

Proof: If the infimum of $\{J(z, v) | (z, v) \in \mathcal{A}_{ad}\}$ equals $+\infty$, the conclusion is straightforward. Without loss of generality, let us consider $\inf\{J(z, v) | (z, v) \in \mathcal{A}_{ad}\} := \delta < +\infty$. From conditions (M1) – (M4), we establish $\delta > -\infty$. By leveraging the concept of infimum, a sequence of feasible pairs $(z^m, v^m) \in \mathcal{A}_{ad}$ emerges, forming a minimizing sequence where $J(z^m, v^m) \rightarrow \delta$ as $m \rightarrow +\infty$, assuming $(z^m, v^m) \in \mathcal{A}_{ad}$ as minimizing sequence. We know $L_p([0, T]; U)$ is a reflexive separable Banach space and $\{v^m\}$ is a bounded subset of $L_p([0, T]; U)$ and also $\{v^m\} \subseteq U_{ad}$: where $m \in \mathbb{N}$, so there is relabeled sequence $\{v^m\}$ and $v^0 \in L_p([0, T]; U)$ such that $v^m \rightarrow v^0$ (weakly converges as $m \rightarrow +\infty$) in $L_p([0, T]; U)$. Knowing that $U_{ad} \subset L_p([0, T]; U)$ is bounded, closed and convex, Mazur's lemma allows us to affirm that $v^0 \in U_{ad}$. Let us now assume that the sequence of solutions of system (2) is given by $\{z^m\}$, which corresponds to the sequence of controls $\{v^m\}$, that is

$$z^m(\varsigma) = -\frac{\partial}{\partial \varsigma} \mathcal{V}(\varsigma, 0)z_0 + \mathcal{V}(\varsigma, 0)z_1 + \int_0^\varsigma \mathcal{V}(\varsigma, \omega) \{Bv^m(\omega) + F(\omega, z^m(\omega))\} d\omega$$

$$+ \int_0^\varsigma \mathcal{V}(\varsigma, \omega) \sigma(\omega, z^m(\omega)) dW(\omega).$$

By Lemma 1, it is clear that $\exists \delta > 0$ such that

$$\mathbb{E} \|z^m\|^2 \leq \delta, \quad m = 0, 1, 2, 3, 4, \dots,$$

Given the control $v^0 \in U_{ad}$, let z^0 be the mild solution for system (2),

$$\begin{aligned} z^0(\varsigma) &= -\frac{\partial}{\partial \varsigma} \mathcal{V}(\varsigma, 0) z_0 + \mathcal{V}(\varsigma, 0) z_1 + \int_0^\varsigma \mathcal{V}(\varsigma, \omega) \{Bv^0(\omega) + F(\omega, z^0(\omega))\} d\omega \\ &+ \int_0^\varsigma \mathcal{V}(\varsigma, \omega) \sigma(\omega, z^0(\omega)) dW(\omega). \end{aligned}$$

By using the Cauchy-Schwarz inequality, Hölder inequality, and condition (M3) for every $\varsigma \in [0, T]$, we obtain

$$\begin{aligned} \mathbb{E} \|z^m(\varsigma) - z^0(\varsigma)\|^2 &\leq 3\mathbb{E} \left\| \int_0^\varsigma \mathcal{V}(\varsigma, \omega) [B(\omega)v^m(\omega) - B(\omega)v^0(\omega)] d\omega \right\|^2 \\ &+ 3\mathbb{E} \left\| \int_0^\varsigma \mathcal{V}(\varsigma, \omega) [F(\omega, z^m(\omega)) - F(\omega, z^0(\omega))] d\omega \right\|^2 \\ &+ 3\mathbb{E} \left\| \int_0^\varsigma \mathcal{V}(\varsigma, \omega) [\sigma(\omega, z^m(\omega)) - \sigma(\omega, z^0(\omega))] dW(\omega) \right\|^2 \\ &\leq 3K^2T \left(\int_0^\varsigma \|B(\omega)v^m(\omega) - B(\omega)v^0(\omega)\|^p d\omega \right)^{\frac{2}{p}} \\ &+ 3K^2T(L_1 + \text{tr}(Q)L_2) \int_0^\varsigma \mathbb{E} \|z^m(\omega) - z^0(\omega)\|^2 d\omega. \end{aligned}$$

Using Gronwall's inequality,

$$\begin{aligned} \mathbb{E} \|z^m(\varsigma) - z^0(\varsigma)\| &\leq K^* \left(\int_0^\varsigma \|B(\omega)v^m(\omega) - B(\omega)v^0(\omega)\|^p d\omega \right)^{\frac{2}{p}} \\ (14) \quad &\leq K^* \|Bv^m - Bv^0\|_{L_p([0, T]; U)}^2. \end{aligned}$$

where K^* is a independent (from m, ς, v) constant.

Considering the strong continuity of B , we can state that

$$(15) \quad \|Bv^m - Bv^0\|_{L_p([0, T]; U)} \rightarrow 0 \text{ as } m \rightarrow \infty$$

We deduce from equations (14) and (15) that

$$\mathbb{E} \|z_\varsigma^m - z_\varsigma^0\|^2 \rightarrow 0 \text{ as } m \rightarrow \infty.$$

$$\Rightarrow \mathbb{E} \|z^m - z^0\|^2 \rightarrow 0 \text{ in } C([0, T]; L_2(\Omega, V)) \text{ as } m \rightarrow \infty.$$

Based on the conditions outlined in (Theorem 2.1, [47]), it is evident that Balder's assumptions hold true under (M1) – (M4). By employing Balder's theorem, we get,

$$(16) \quad (z, v) \rightarrow \mathbb{E} \int_0^\varsigma L(\omega, z(\omega), v(\omega)) d\omega.$$

in the strong topology of $L_1([0, T]; V)$ and weak topology of $L_p([0, T]; U) \subset L_1([0, T], U)$ is sequentially lower semicontinuous. Therefore on $L_p([0, T]; U)$, J exhibits weak lower semicontinuity. Additionally, as per condition (M4), J remains bounded below by $-\infty$. The infimum of J is achieved at $v^0 \in U_{ad}$, where

$$\begin{aligned} \delta &:= \lim_{m \rightarrow \infty} \int_0^\varsigma \mathbb{E} L(\omega, z^m(\omega), v^m(\omega)) d\omega \\ &\geq \int_0^\varsigma \mathbb{E} L(\omega, z^0(\omega), v^0(\omega)) d\omega = J(z^0, v^0) \geq \delta. \end{aligned}$$

This completes the proof.

5 Examples

Assume the second order semilinear stochastic integro differential system:

$$z''(\varsigma) = -\lambda z(\varsigma) + \int_0^\varsigma e^{-(\varsigma-\omega)} \sin(z(\omega)) d\omega + z^3(\varsigma) + v(\varsigma) + \sigma z(\varsigma) dW(\varsigma), \quad \varsigma \in [0, 1],$$

with parameters: $\lambda = 1, \sigma = 0.1$,

initial conditions $z(0) = 1, z'(0) = 0$.

The cost function is:

$$J(z, v) := \mathbb{E} \left\{ \int_0^1 \left[\frac{1}{2} z^2(\varsigma) + \frac{1}{2} v^2(\varsigma) \right] d\varsigma \right\}.$$

by generating the increments for Brownian motion and Euler-Maruyama method to iterate over the time steps to update the state z and its derivative using the Euler-Maruyama method and compute the integral term using numerical integration (trapezoidal rule).

We get the state trajectory $z(t)$ and the control trajectory $v(t)$ are plotted over time.

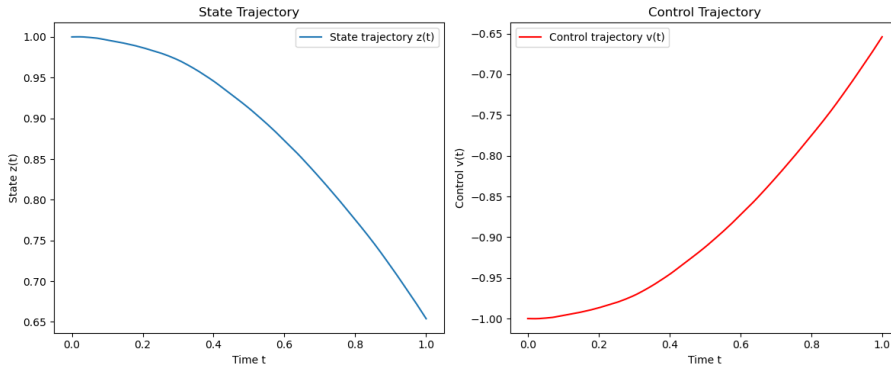


Figure 1: state and control trajectories

Conclusion on Trajectories

State Trajectory $z(t)$:

The state trajectory $z(t)$ shows how the state evolves over time under the influence of the nonlinear dynamics and stochastic perturbations. The trajectory might exhibit oscillatory behavior due to the nonlinear term $z^3(t)$ and the integral term involving $\sin(z(t))$. Additionally, the stochastic component, represented by the Brownian motion term $\sigma z(t)dW(t)$ introduces randomness, causing fluctuations and deviations from a deterministic path.

Control Trajectory $v(t)$:

The control trajectory $v(t)$ in this example is initially set to zero and remains zero throughout the simulation. As a result, it does not directly influence the state trajectory in this particular run. The state trajectory evolves based solely on the initial conditions, system dynamics, and stochastic influences. In a more comprehensive scenario, $v(t)$ would be dynamically adjusted through an optimization process to minimize the cost functional, potentially leading to more controlled and less oscillatory state trajectories.

6 Conclusion

This paper presented a comprehensive framework for the optimal control of second-order semilinear integro-differential stochastic systems using resolvent operators. We established the existence and uniqueness of mild solutions and derived necessary and sufficient optimality conditions. The application of resolvent operator theory effectively managed the system's integral and stochastic components, while the minimizing sequence approach and Mazur's lemma facilitated the construction of the optimal control. Our theoretical results were validated through an illustrative example, demonstrating the practical applicability of our methods. This study provides a robust foundation for further research and practical implementations in complex stochastic control systems.

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